Functors

October 15, 2019

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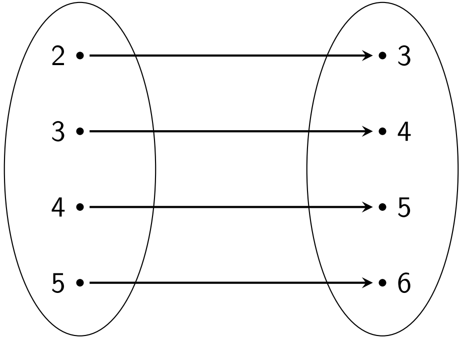

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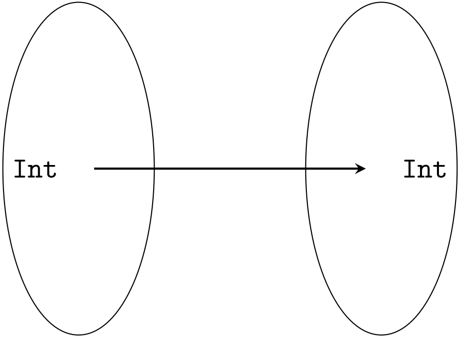

- `succ :: Int -> Int`

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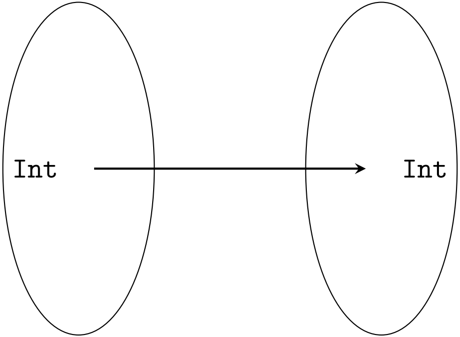

- `succ :: Enum a => a -> a`

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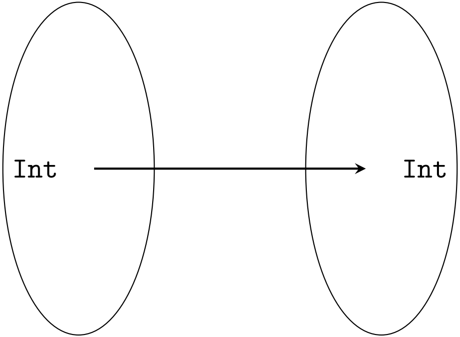

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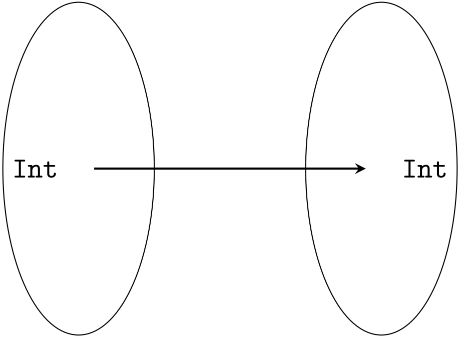

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- `succ` is a function from `Int`s to `Int`s.

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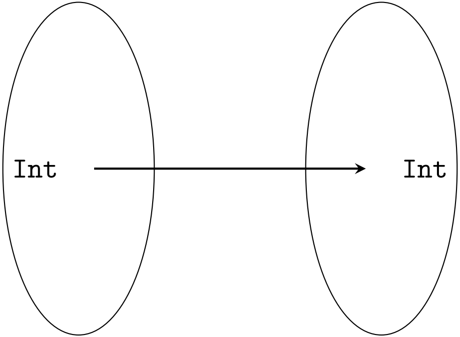

- `succ :: Int -> Int`
- `succ` is a morphism from `Ints` to `Ints`.

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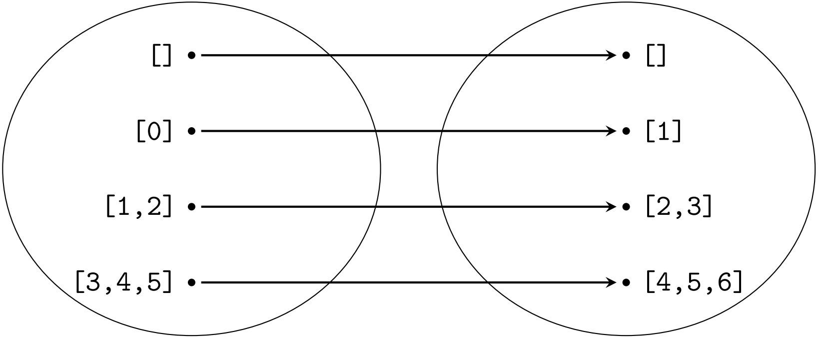

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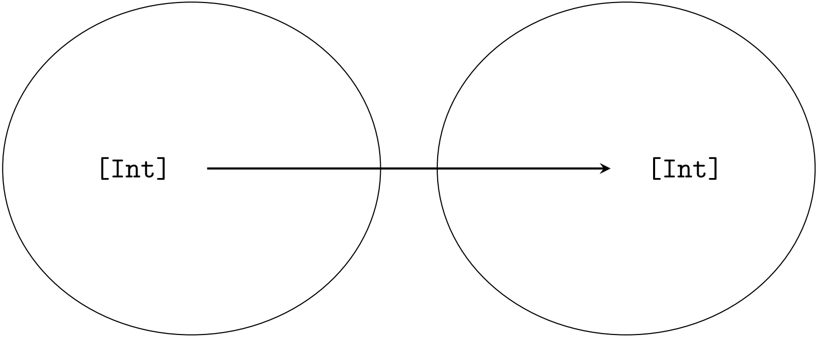

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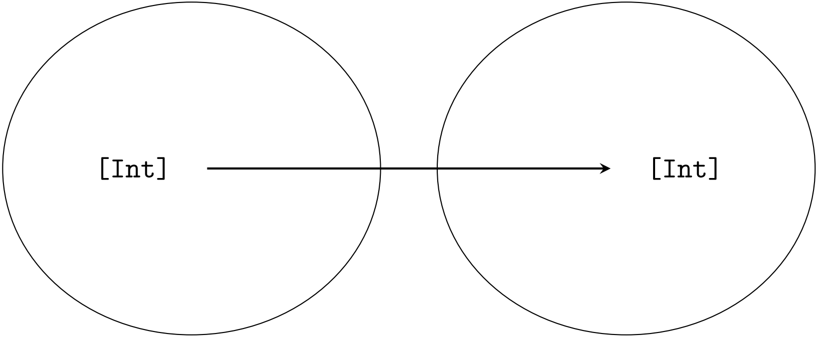

- `map succ :: [Int] -> [Int]`

Functors

- Wikipedia: Let C and D be categories. A functor F from C to D is a mapping that
  - associates to each object X in C an object $F(X)$ in D,
  - associates to each morphism $f\colon X \to Y$ in C a morphism $F(f)\colon F(X) \to F(Y)$ in D such that the following two conditions hold:
    - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object X in C,
    - $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in C.

  That is, functors must preserve identity morphisms and composition of morphisms.

Functors

- Let $C = D =$ `Hask`.
  - `Hask` = the category of Haskell types
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That is, functors must preserve identity morphisms and composition of morphisms.(Haskell will not do this for you—you have to do it yourself)

Lists are Functors

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  instance Functor [] where
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  - `Maybe`
  - `IO`
  - Functions (`((->) r)`)

Functors

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  - That implement maps that lift normal functions (of type `a -> b`) to functions over boxes (of type `F a -> F b`)

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  - That implement maps that lift normal functions (of type `a -> b`) to functions over context (of type `F a -> F b`)
    - `IO`: input/output
    - `Maybe`: possible failure
    - `[]`: nondeterminism