Modal and Temporal Logic

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October 26, 2022

Announcements

- Paper presentation schedule posted on LATTE
  - Starting 11/21
- For next Monday
  - Read van Eijck and Unger Chapter 11
- For 11/2
  - HW3 due
- For 11/9
  - Final Project Idea due

Paper Presentation Guidelines

- Groups should aim for around 20 minutes for summary and analysis, and around 5 minutes for questions and discussion
- Presentations should cover the following themes
  - Describe the problem, and why it is interesting/important
    - What dimensions of meaning are the authors interested in (e.g., expression meaning vs. speaker meaning, meaning as truth vs. meaning as use, etc.)?
  - How do the authors try to solve the problem?
    - Methods, data, evaluation, etc.
  - What are the results and conclusions?
    - For someone interested in (a) different dimension(s) of meaning than the authors, what lessons can they learn from the paper?

Final Project

- In small groups, students will investigate a topic of their choice related to computational semantics, culminating in a short write-up and presentation during finals week
  - By 11/9, please prepare a short document (one per group, 1/2–1 page or so, in PDF format) containing:

Final Project

1. Names of group members
   - Groups of 2 or 3 preferred
     - Groups of 1 or 4 also possible, with permission from me
     - You may (but don't have to) remain in the same groups as for the paper presentations

Final Project

2. Topic you want to investigate
   - Describe the problem, and why it is interesting/important
   - Also try to include at least a little bit of background/related work
   - Possible topics include, but are not limited to
     - Extensions of things covered in class/the book
     - Questions that came up when reading a paper (for the paper presentations, or otherwise)
     - etc.

Final Project

3. How you will investigate it
   - Include methods, data, and evaluation (if applicable)
     - This can look very different for different projects
   - Projects should involve a "substantial" programming component
     - Does not have to be in Haskell
     - If you are unsure if your programming component is "substantial", ask!

Final Project

- Important dates and deadlines
  - 11/9: Idea due
  - 12/7: Progress report/"rough draft" due
    - Don't worry, you don't have to have your project done by then!
  - 12/14 6-9pm: Presentations (location TBD)
  - 12/19: Code and write-up due
    - Expected write-up length: 2–4 pages, single-spaced/formatted according to a style file to be provided

Final Project

- Note that we do not expect you to fully solve the problem you are investigating
  - Toy models, or partial/inconclusive results are good!
  - The purpose of the final project is to explore a topic you are interested in using the knowledge and skills you've learned during the semester

Today's Plan

- Modal Logic
  - Syntax
  - Semantics
- Temporal Logic
  - Syntax
  - Semantics

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- (we'll see how far we get...)

Intensional Constructs

- "Whether or not a statement is true in some model depends on what is the actual world in that model.
  - The actual world is the world where we evaluate.
- To check whether a statement is necessarily true in an intensional model, we have to check whether it is true in all possible worlds.
- A statement is contingently true if it is true, but it ain't necessarily so: there exists a world where that statement is false."

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```
iJudgement :: Judgement -> IBool
iJudgement (IsTrue s) = \ i -> iSent s i
iJudgement (IsNec s) = \ i ->
  all (\j -> iSent s j) worlds
iJudgement (IsCont s) = \ i ->
  iSent s i && not (all (\j -> iSent s j) worlds)
```

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- Modal logic is the logic of possible worlds

Modal Logic

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Modal Logic

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- Atomic propositions
  - Typically indicated by lower case letters p, q, r, etc., possibly with indices
  - Represent the meanings of certain declarative sentences
    - Specifically, those that cannot be decomposed into other atomic propositions and logical connectives
  - For example, let:
    - p be "It rains"
    - q be "The sun is shining"
    - r be "There will be a rainbow"

Modal Logic

- Formulas of propositional logic
  - Atomic propositions are formulas
  - Let F_1 and F_2 be formulas. Then the following are also formulas:
    - Negation: $\neg F_1$ ("not F_1")
    - Conjunction: $(F_1 \wedge F_2)$ ("F_1 and F_2")
    - Disjunction: $(F_1 \vee F_2)$ ("F_1 or F_2")
    - Implication (or conditional): $(F_1 \rightarrow F_2)$ ("if F_1 then F_2")
    - Equivalence (or biconditional): $(F_1 \leftrightarrow F_2)$ ("F_1 if and only if F_2")
- For example, the sentence "If it rains and the sun is shining, then there will be a rainbow" can be represented as the propositional formula $(p \wedge q) \rightarrow r$

Modal Logic

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Modal Logic

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  - Let F be a formula. Then the following are also formulas:
    - Box: $\Box F$ ("necessarily F")
    - Diamond: $\Diamond F$ ("possibly F")

Semantics of Modal Logic

- **Definition 5.3** [Huth and Ryan] A model $\mathcal{M}$ of basic modal logic is specified by three things:
  1. A set W, whose elements are called worlds;
  2. A relation R on W ($R \subseteq W \times W$), called the accessibility relation;
  3. A function $L : W \to \mathcal{P}(\mathtt{Atoms})$, called the labelling function.

  We write $R(x, y)$ to denote that (x, y) is in R.
- "These models are often called Kripke models, in honour of S. Kripke who invented them and worked extensively in modal logic in the 1950s and 1960s."

Semantics of Modal Logic

- Huth and Ryan: "Intuitively, $w \in W$ stands for a possible world and $R(w, w')$ means that w' is a world accessible from world w.
  - The actual nature of that relationship depends on what we intend to model."
- L is a function from possible worlds to sets of atomic propositions that are true in those worlds

Semantics of Modal Logic

- Graphical notation
  - "The set W is drawn as a set of circles, with arrows between them showing the relation R. Within each circle is the value of the labelling function in that world."

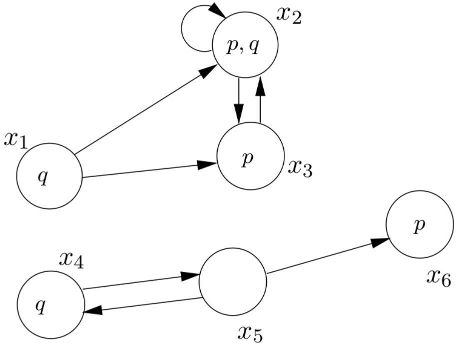

Figure 5.3. A Kripke model.

Semantics of Modal Logic

- The truth of an atomic proposition in a world is determined by the value of the labelling function in the world
- For other formulas:
  - $\neg F_1$ is true in world x iff F_1 is false in x
  - $F_1 \wedge F_2$ is true in world x iff F_1 is true and F_2 is true in x
  - $F_1 \vee F_2$ is true in world x iff F_1 is true or F_2 is true in x
  - $F_1 \rightarrow F_2$ is true in world x iff F_1 is false or F_2 is true in x
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  - $\Box F_1$ is true in world x iff for all $y \in W$ such that $R(x, y)$, F_1 is true in y
  - $\Diamond F_1$ is true in world x iff there exists a $y \in W$ such that $R(x, y)$ and F_1 is true in y
- We can write $x \Vdash F$ when F is true in world x

Semantics of Modal Logic

- **Definition 5.7**
  1. We say that a set of formulas Γ of basic modal logic semantically entails a formula ψ of basic modal logic if, in any world x of any model $\mathcal{M} = (W, R, L)$, we have $x \Vdash \psi$ whenever $x \Vdash \phi$ for all $\phi \in \Gamma$. In that case, we say that $\Gamma \models \psi$ holds.
  2. We say that ϕ and ψ are semantically equivalent if $\phi \models \psi$ and $\psi \models \phi$ hold. We denote this by $\phi \equiv \psi$.
- **Definition 5.8** A formula ϕ of basic modal logic is said to be valid if it is true in every world of every model, i.e. iff $\models \phi$ holds.

Semantics of Modal Logic

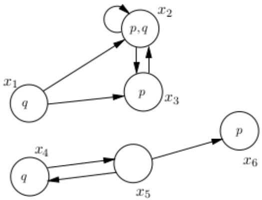

Figure 5.3. A Kripke model.

Examples 5.6 Consider the Kripke model of Figure 5.3. We have:

- $x_1 \Vdash q$, since $q \in L(x_1)$.
- $x_1 \Vdash \Diamond q$, for there is a world accessible from x_1 (namely, x_2) which satisfies q. In mathematical notation: $R(x_1, x_2)$ and $x_2 \Vdash q$.
- $x_1 \nVdash \Box q$, however. This is because $x_1 \Vdash \Box q$ says that all worlds accessible from x_1 (i.e. x_2 and x_3) satisfy q; but x_3 does not.
- $x_5 \nVdash \Box p$ and $x_5 \nVdash \Box q$. Moreover, $x_5 \nVdash \Box p \vee \Box q$. However, $x_5 \Vdash \Box(p \vee q)$. To see these facts, note that the worlds accessible from x_5 are x_4 and x_6. Since $x_4 \nVdash p$, we have $x_5 \nVdash \Box p$; and since $x_6 \nVdash q$, we have $x_5 \nVdash \Box q$. Therefore, we get that $x_5 \nVdash \Box p \vee \Box q$. However, $x_5 \Vdash \Box(p \vee q)$ holds because, in each of x_4 and x_6, we find p or q.
- The worlds which satisfy $\Box p \rightarrow p$ are x_2, x_3, x_4, x_5 and x_6; for x_2, x_3 and x_6 this is so since they already satisfy p; for x_4 this is true since it does not satisfy $\Box p$ – we have $R(x_4, x_5)$ and x_5 does not satisfy p; a similar reason applies to x_5. As for x_1, it cannot satisfy $\Box p \rightarrow p$ since it satisfies $\Box p$ but not p itself.

Semantics of Modal Logic

- "Worlds like x_6 that have no world accessible to them deserve special attention in modal logic.
  - Observe that $x_6 \nVdash \Diamond\phi$, no matter what ϕ is, because $\Diamond\phi$ says 'there is an accessible world which satisfies ϕ.' In particular, 'there is an accessible world,' which in the case of x_6 there is not."
    - Like predicate logic $\exists$
  - "No matter what ϕ is, we find that $x_6 \Vdash \Box\phi$ holds. That is because $x_6 \Vdash \Box\phi$ says that ϕ is true in all worlds accessible from x_6. There are no such worlds, so ϕ is vacuously true in all of them: there is simply nothing to check."
    - Like predicate logic $\forall$

Temporal Logic

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  - Special case of modal logic
  - Time indices are represented as possible worlds/states

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- Two major kinds of temporal logics
  - "Linear-time logics think of time as a set of paths, where a path is a sequence of time instances."
    - e.g., Linear-time Temporal Logic (LTL), etc.
  - "Branching-time logics represent time as a tree, rooted at the present moment and branching out into the future."
    - e.g., Computation Tree Logic (CTL), etc.

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- The logic of HW3 is a linear-time logic with past and future temporal operators

Temporal Logic

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Temporal Logic

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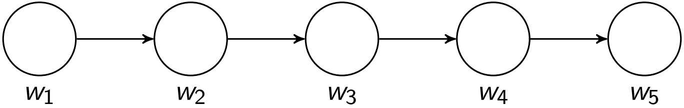
  - Formulas of propositional logic are formulas of temporal logic
  - Let ϕ be a formula. Then the following are also formulas:
    - "H": $H\phi$ ("at every time in the past, ϕ")
    - Past: $P\phi$ ("at some time in the past, ϕ")
    - Future: $F\phi$ ("at some time in the future, ϕ")
    - Global: $G\phi$ ("at every time in the future, ϕ")

Semantics of Temporal Logic

- A path in a model $\mathcal{M} = (W, R, L)$ is a sequence of worlds $w_1, w_2, w_3 \ldots$ in W such that, for each $t \geq 1$, $R(w_t, w_{t+1})$



Semantics of Temporal Logic

- The truth of an atomic proposition at a time is determined by the value of the labelling function at the time
- For other formulas, given a path $w_1, w_2, w_3, \ldots$:
  - $\neg\phi$ is true at time t iff ϕ is false at t
  - $\phi \wedge \psi$ is true at time t iff ϕ is true and ψ is true at t
  - $\phi \vee \psi$ is true at time t iff ϕ is true or ψ is true at t
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  - $\phi \leftrightarrow \psi$ is true at time t iff ϕ and ψ have the same truth value at t
  - $H\phi$ is true at time t iff for all $i < t$, ϕ is true at time i
  - $P\phi$ is true at time t iff there exists an $i < t$ such that ϕ is true at time i
  - $F\phi$ is true at time t iff there exists an $i \geq t$ such that ϕ is true at time i
  - $G\phi$ is true at time t iff for all $i \geq t$, ϕ is true at time i
- Notice that the future includes the present!