Haskell — Higher-Order Functions

Higher-Order Functions

- A function is Higher-Order if it takes a function as an argument or returns one as its result.
- Higher-order function aren't weird; the differentiation operation from high-school calculus is higher-order:

```
deriv ::  (Float->Float)->Float->Float
deriv f x = (f(x+dx) - f x)/0.0001
```

- Many recursive functions share a similar structure. We can capture such "recursive patterns" in a higher-order function.
- We can often avoid the use of explicit recursion by using higher-order functions. This leads to functions that are shorter, and easier to read and maintain.

Currying Revisited

- We have already seen a number of higher-order functions. In fact, any curried function is higher-order. Why? Well, when a curried function is applied to one of its arguments it returns a new function as the result.

Uh, what was this currying thing?

- A curried function does not have to be applied to all its arguments at once. We can supply some of the arguments, thereby creating a new specialized function. This function can, for example, be passed as argument to a higher-order function.

CurRying Revisited...

How is a curried function defined?

- A curried function of n arguments (of types $\texttt{t}_1, \texttt{t}_2, \cdots, \texttt{t}_n$) that returns a value of type `t` is defined like this:

$$\texttt{fun :: } \texttt{t}_1 \texttt{ -> } \texttt{t}_2 \texttt{ -> } \cdots \texttt{ -> } \texttt{t}_n \texttt{ -> t}$$

- This is sort of like defining n different functions (one for each `->`). In fact, we could define these functions explicitly, but that would be tedious:

$$\texttt{fun}_1 \texttt{ :: } \texttt{t}_2 \texttt{ -> } \cdots \texttt{ -> } \texttt{t}_n \texttt{ -> t}$$
$$\texttt{fun}_1 \ \texttt{a}_2 \cdots \texttt{a}_n \texttt{ = } \cdots$$

$$\texttt{fun}_2 \texttt{ :: } \texttt{t}_3 \texttt{ -> } \cdots \texttt{ -> } \texttt{t}_n \texttt{ -> t}$$
$$\texttt{fun}_2 \ \texttt{a}_3 \cdots \texttt{a}_n \texttt{ = } \cdots$$

Currying Revisited. . .

Duh, how about an example?

- Certainly. Lets define a recursive function `get_nth n xs` which returns the `n`:th element from the list `xs`:

```
get_nth 1 (x:_) = x
get_nth n (_:xs) = get_nth (n-1) xs
```

```
get_nth 10 "Bartholomew" ⇒ 'e'
```

- Now, let's use `get_nth` to define functions `get_second`, `get_third`, `get_fourth`, and `get_fifth`, without using explicit recursion:

```
get_second = get_nth 2 | get_fourth = get_nth 4
get_third =  get_nth 3 | get_fifth =  get_nth 5
```

Currying Revisited...

```
get_fifth "Bartholomew" ⇒ 'h'

map (get_nth 3)
   ["mob","sea","tar","bat"] ⇒
   "bart"
```

So, what's the type of `get_second`?

- Remember the Rule of Cancellation?
- The type of `get_nth` is `Int -> [a] -> a`.
- `get_second` applies `get_nth` to one argument. So, to get the type of `get_second` we need to cancel `get_nth`'s first type: ~~`Int`~~ `-> [a] -> a` ≡ `[a] -> a`.

Patterns of Computation

Mappings

- Apply a function f to the elements of a list L to make a new list L'. Example: Double the elements of an integer list.

Selections

- Extract those elements from a list L that satisfy a predicate p into a new list L'. Example: Extract the even elements from an integer list.

Folds

- Combine the elements of a list L into a single element using a binary function f. Example: Sum up the elements in an integer list.

The map Function

- map takes two arguments, a function and a list. map creates a new list by applying the function to each element of the input list.
- map's first argument is a function of type a -> b. The second argument is a list of type [a]. The result is a list of type [b].

```
map ::  (a -> b) -> [a] -> [b]
map f [ ]          = [ ]
map f (x:xs)       = f x :  map f xs
```

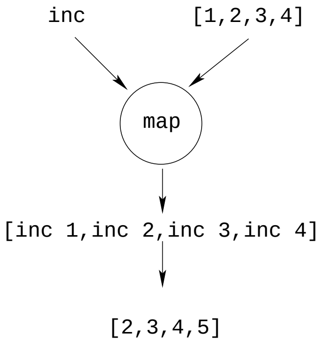
- We can check the type of an object using the :type command. Example: :type map.

The map Function...

```
map ::  (a -> b) -> [a] -> [b]
map f [ ]    = [ ]
map f (x:xs)= f x :  map f xs

inc x = x + 1

map inc [1,2,3,4] ⇒ [2,3,4,5]
```



The map Function...

```
map ::  (a -> b) -> [a] -> [b]
map f [ ]          = [ ]
map f (x:xs)       = f x :  map f xs
```

map f [ ] = [ ] means: "The result of applying the function `f` to the elements of an empty list is the empty list."

map f (x:xs) = f x : map f xs means: "applying `f` to the list `(x:xs)` is the same as applying `f` to `x` (the first element of the list), then applying `f` to the list `xs`, and then combining the results."

The map Function...

Simulation:

```
map square [5,6] ⇒
   square 5 :  map square [6] ⇒
   25 :  map square [6] ⇒
      25 :  (square 6 :  map square [ ]) ⇒
      25 :  (36 :  map square [ ]) ⇒
         25 :  (36 :  [ ]) ⇒
      25 :  [36] ⇒
   [25,36]
```

The `filter` Function

- Filter takes a predicate p and a list L as arguments. It returns a list L' consisting of those elements from L that satisfy p.
- The predicate p should have the type `a -> Bool`, where a is the type of the list elements.

Examples:

```
filter even [1..10] ⇒ [2,4,6,8,10]
filter even (map square [2..5]) ⇒
   filter even [4,9,16,25] ⇒ [4,16]
filter gt10 [2,5,9,11,23,114]
   where gt10 x = x > 10 ⇒ [11,23,114]
```

The `filter` Function...

- We can define `filter` using either recursion or list comprehension.

Using recursion:

```
filter ::  (a -> Bool) -> [a] -> [a]
filter _ [] = []
filter p (x:xs)
   | p x       = x :  filter p xs
   | otherwise = filter p xs
```

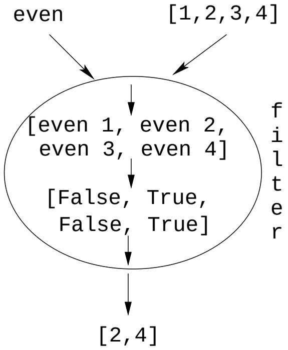
Using list comprehension:

```
filter ::  (a -> Bool) -> [a] -> [a]
filter p xs = [x | x <- xs, p x]
```

The filter Function...

```
filter ::  (a->Bool)->[a]->[a]
filter _ [] = []
filter p (x:xs)
  | p x = x :  filter p xs
  | otherwise = filter p xs
```

filter even [1,2,3,4] ⇒ [2,4]



The filter Function...

- doublePos doubles the positive integers in a list.

```
getEven ::  [Int] -> [Int]
getEven xs = filter even xs

doublePos ::  [Int] -> [Int]
doublePos xs = map dbl (filter pos xs)
               where dbl x = 2 * x
                     pos x = x > 0
```

Simulations:

```
getEven [1,2,3] ⇒ [2]

doublePos [1,2,3,4] ⇒
   map dbl (filter pos [1,2,3,4]) ⇒
   map dbl [2,4] ⇒ [4,8]
```

fold Functions

- A common operation is to combine the elements of a list into one element. Such operations are called reductions or accumulations.

Examples:

```
sum [1,2,3,4,5] ≡
      (1 + (2 + (3 + (4 + (5 + 0))))) ⇒ 15
concat ["H","i","!"] ≡
      ("H" ++ ("i" ++ ("!" ++ ""))) ⇒ "Hi!"
```

- Notice how similar these operations are. They both combine the elements in a list using some binary operator (+, ++), starting out with a "seed" value (0, "").

`fold` Functions...

- Haskell provides a function `foldr` ("fold right") which captures this pattern of computation.
- `foldr` takes three arguments: a function, a seed value, and a list.

Examples:

```
foldr (+) 0 [1,2,3,4,5] ⇒ 15
foldr (++) "" ["H","i","!"] ⇒ "Hi!"
```

foldr:

```
foldr ::  (a->b->b) -> b -> [a] -> b
foldr f z [ ]      = z
foldr f z (x:xs)   = f x (foldr f z xs)
```

fold Functions...

- Note how the fold process is started by combining the last element x_n with z. Hence the name seed.

$$\texttt{foldr}(\oplus)z[x_1 \cdots x_n] = (x_1 \oplus (x_2 \oplus (\cdots (x_n \oplus z))))$$

- Several functions in the standard prelude are defined using foldr:

```
and,or ::  [Bool] -> Bool
and xs = foldr (&&) True xs
or xs = foldr (||) False xs
```

```
?  or [True,False,False] ⇒
   foldr (||) False [True,False,False] ⇒
   True || (False || (False || False)) ⇒ True
```

`fold` Functions...

- Remember that `foldr` binds from the right:

```
foldr (+) 0 [1,2,3] ⇒ (1+(2+(3+0)))
```

- There is another function `foldl` that binds from the left:

```
foldl (+) 0 [1,2,3] ⇒ (((0+1)+2)+3)
```

- In general:

$$\texttt{foldl}(\oplus)\texttt{z}[\texttt{x}_1 \cdots \texttt{x}_n] = (((\texttt{z} \oplus \texttt{x}_1) \oplus \texttt{x}_2) \oplus \cdots \oplus \texttt{x}_n)$$

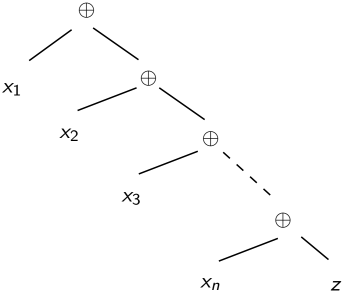
`fold` Functions...

- In the case of `(+)` and many other functions

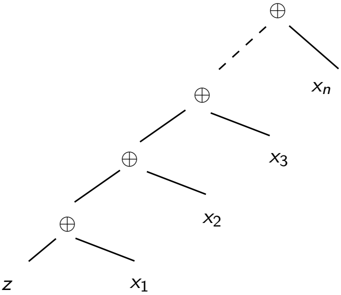
$$\texttt{foldl}(\oplus)\texttt{z}[\texttt{x}_1 \cdots \texttt{x}_n] \;=\; \texttt{foldr}(\oplus)\texttt{z}[\texttt{x}_1 \cdots \texttt{x}_n]$$

- However, one version may be more efficient than the other.

fold Functions...



foldr $\oplus$ z $[x_1 \cdots x_n]$



foldl $\oplus$ z $[x_1 \cdots x_n]$

Operator Sections

- We've already seen that it is possible to use operators to construct new functions:

  `(*2)` – function that doubles its argument

  `(>2)` – function that returns `True` for numbers > 2.

- Such partially applied operators are known as operator sections. There are two kinds:

`(op a) b = b op a`

```
(*2) 4              = 4 * 2  = 8
(>2) 4              = 4 > 2  = True
(++ "\n") "Bart"    = "Bart" ++ "\n"
```

Operator Sections...

(a op) b = a op b

```
(3:)  [1,2] = 3 :  [1,2]= [3,1,2]
(0<) 5      = 0 < 5      = True
(1/) 5      = 1/5
```

Examples:

- `(+1)` – The successor function.
- `(/2)` – The halving function.
- `(:[])` – The function that turns an element into a singleton list.

More Examples:

```
? filter (0<) (map (+1) [-2,-1,0,1])
  [1,2]
```

takeWhile & dropWhile

- We've looked at the list-breaking functions drop & take:

```
take 2 ['a','b','c'] ⇒ ['a','b']
drop 2 ['a','b','c'] ⇒ ['c']
```

- takeWhile and dropWhile are higher-order list-breaking functions. They take/drop elements from a list while a predicate is true.

```
takeWhile even [2,4,6,5,7,4,1] ⇒
   [2,4,6]
dropWhile even [2,4,6,5,7,4,1] ⇒
   [5,7,4,1]
```

takeWhile & dropWhile...

```
takeWhile ::  (a->Bool) -> [a] -> [a]
takeWhile p [ ] = [ ]
takeWhile p (x:xs)
   | p x        = x :  takeWhile p xs
   | otherwise = [ ]

dropWhile ::  (a->Bool) -> [a] -> [a]
dropWhile p [ ] = [ ]
dropWhile p (x:xs)
   | p x        = dropWhile p xs
   | otherwise = x:xs
```

takeWhile & dropWhile...

- Remove initial/final blanks from a string:

```
dropWhile ((==) '␣') "␣␣␣Hi!" ⇒
    "Hi!"

takeWhile ((/=) '␣') "Hi!␣␣␣" ⇒
    "Hi!"
```

Summary

- Higher-order functions take functions as arguments, or return a function as the result.
- We can form a new function by applying a curried function to some (but not all) of its arguments. This is called partial application.
- Operator sections are partially applied infix operators.

Summary...

- The standard prelude contains many useful higher-order functions:

  map f xs creates a new list by applying the function `f` to every element of a list `xs`.

  filter p xs creates a new list by selecting only those elements from `xs` that satisfy the predicate `p` (i.e. `(p x)` should return `True`).

  foldr f z xs reduces a list `xs` down to one element, by applying the binary function `f` to successive elements, starting from the right.

  scanl/scanr f z xs perform the same functions as `foldr/foldl`, but instead of returning only the ultimate value they return a list of all intermediate results.

Exercise

Exercise (a):

- Define the `map` function using a list comprehension.

Template:

```
map f x = [ ··· | ··· ]
```

Exercise (b):

- Use `map` to define a function `lengthall xss` which takes a list of strings `xss` as argument and returns a list of their lengths as result.

Examples:

```
?  lengthall ["Ay", "Caramba!"]
   [2,8]
```

Exercise

1. Give a accumulative recursive definition of `foldl`.
2. Define the `minimum xs` function using `foldr`.
3. Define a function `sumsq n` that returns the sum of the squares of the numbers $[1 \cdots n]$. Use `map` and `foldr`.
4. What does the function `mystery` below do?

```
mystery xs =
   foldr (++) [] (map sing xs)
sing x = [x]
```

Examples:

```
minimum [3,4,1,5,6,3] ⇒ 1
```

Exercise...

- Define a function `zipp f xs ys` that takes a function `f` and two lists $xs=[x_1,\cdots,x_n]$ and $ys=[y_1,\cdots,y_n]$ as argument, and returns the list $[f\ x_1\ y_1,\cdots,f\ x_n\ y_n]$ as result.
- If the lists are of unequal length, an error should be returned.

Examples:

```
zipp (+) [1,2,3] [4,5,6] ⇒ [5,7,9]

zipp (==) [1,2,3] [4,2,2] ⇒ [False,True,False]

zipp (==) [1,2,3] [4,2] ⇒ ERROR
```

Exercise

- Define a function `filterFirst p xs` that removes the first element of `xs` that does not have the property `p`.

Example:

```
filterFirst even [2,4,6,5,6,8,7] ⇒
    [2,4,6,6,8,7]
```

- Use `filterFirst` to define a function `filterLast p xs` that removes the last occurence of an element of `xs` without the property `p`.

Example:

```
filterLast even [2,4,6,5,6,8,7] ⇒
    [2,4,6,5,6,8]
```