# Functors

September 20, 2024

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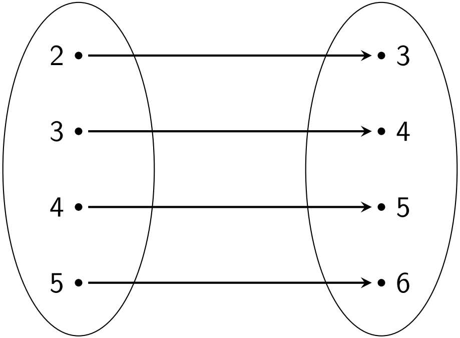

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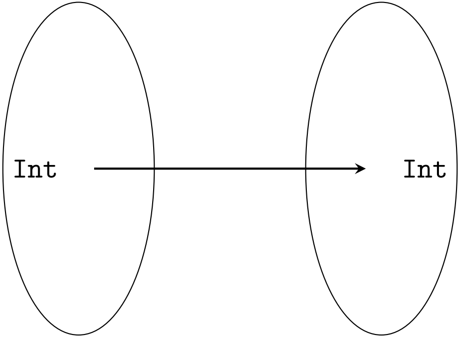


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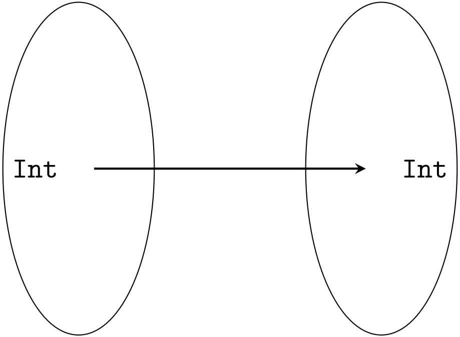


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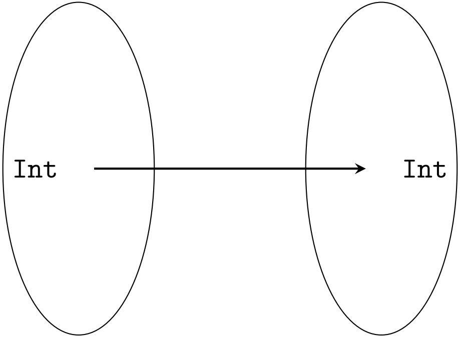


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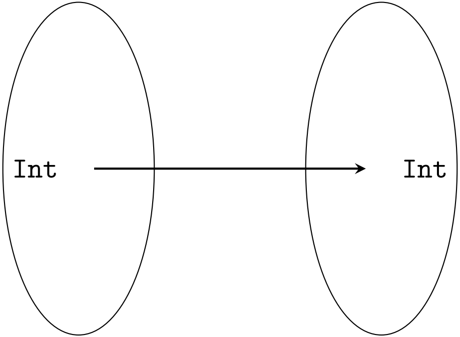


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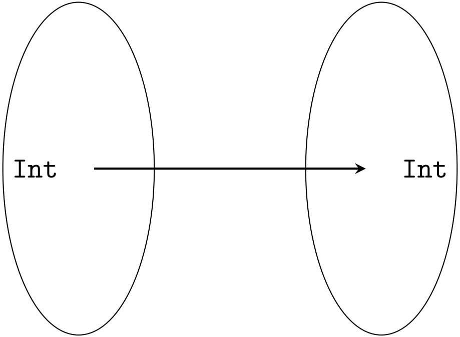


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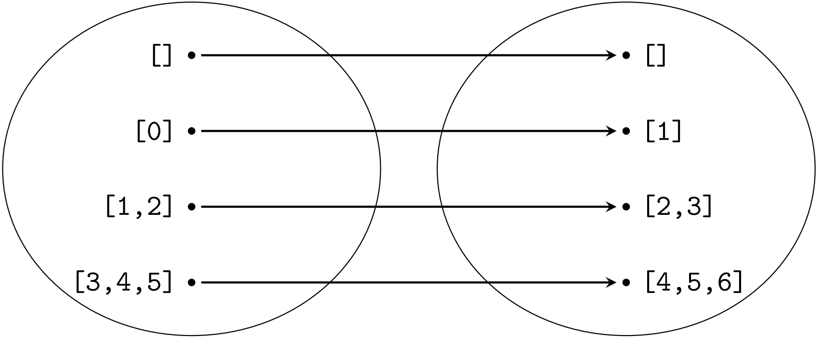

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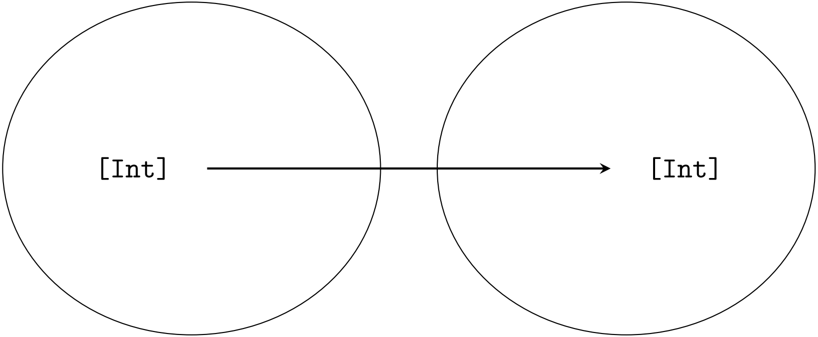

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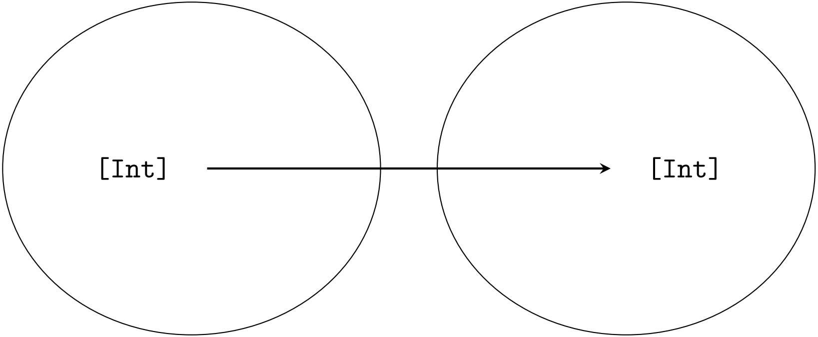


- `map succ :: [Int] -> [Int]`

# Functors

- Wikipedia: Let $C$ and $D$ be categories. A functor $F$ from $C$ to $D$ is a mapping that
  - associates to each object $X$ in $C$ an object $F(X)$ in $D$,
  - associates to each morphism $f\colon X \to Y$ in $C$ a morphism $F(f)\colon F(X) \to F(Y)$ in $D$ such that the following two conditions hold:
    - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X$ in $C$,
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  That is, functors must preserve identity morphisms and composition of morphisms.

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  - $\texttt{Hask}$ = the category of Haskell types
    - category = objects + morphisms
      - objects = types
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That is, functors must preserve identity morphisms and composition of morphisms. (Haskell will not do this for you—you have to do it yourself)

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  - Functions (`((->) r`)

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  - That implement maps that lift normal functions (of type `a -> b`) to functions over context (of type `F a -> F b`)
    - `IO`: input/output
    - `Maybe`: possible failure
    - `[]`: nondeterminism